

Lorentz Invariant Vacuum Solutions in General Relativity

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Abstract

All Lorentz invariant solutions of vacuum Einstein's equations are found. It is proved that these solutions describe space-times of constant curvature.

Derivation of exact solutions of Einstein's equations is one of the main problems in general relativity. Many classes of exact solutions are given in [1]. As a rule, solutions are derived under the assumption of some symmetry because Einstein's equations are very complicated. This simplifies the system of equations and opens a possibility to look for solutions.

De Sitter and anti-de Sitter solutions [2, 3] were among the first exact cosmological solutions of vacuum equations with a cosmological constant. The De Sitter solution describes the space-time of constant curvature and is invariant with respect to the action of the Lorentz group $\mathbb{SO}(1, 4)$. The Anti-de Sitter solution corresponds to the constant curvature space and is invariant with respect to the group $\mathbb{SO}(2, 3)$. Both symmetry groups contain the Lorentz subgroup $\mathbb{SO}(1, 3)$ of lower dimension. In the present paper, we prove that all vacuum solutions of Einstein's equations invariant with respect to the action of the Lorentz group $\mathbb{SO}(1, 3)$ describe space-times of constant curvature and therefore reduce either to de Sitter or to anti-de Sitter solutions depending on the sign of the cosmological constant.

Consider the Minkowski space-time $\mathbb{R}^{1, n-1}$ of arbitrary dimension n . The Lorentz metric $\eta_{\alpha\beta} := \text{diag}(+ - \dots -)$ is given in the Cartesian coordinate system x^α , $\alpha = 0, 1, \dots, n-1$. This metric is invariant with respect to the Poincaré group and, in particular, with respect to the Lorentz group. Corresponding Killing vector fields are

$$K_{\epsilon\delta} = \frac{1}{2} (x_\delta \partial_\epsilon - x_\epsilon \partial_\delta) = \frac{1}{2} (x_\delta \delta_\epsilon^\gamma - x_\epsilon \delta_\delta^\gamma) \partial_\gamma, \quad (1)$$

where the indices ϵ and δ number $n(n-1)/2$ Killing vectors and $x_\alpha := x^\beta \eta_{\beta\alpha}$. Let another metric $g_{\alpha\beta}(x)$ be given in the Minkowski space-time $\mathbb{R}^{1, n-1}$. We pose the following problem: find all metrics $g_{\alpha\beta}$ invariant under the action of the Lorentz group $\mathbb{SO}(1, n-1)$.

The equations

$$\nabla_\alpha K_\beta + \nabla_\beta K_\alpha = 0$$

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for the Killing vector fields (1) take the form

$$g_{\alpha\epsilon}\eta_{\beta\delta} - g_{\alpha\delta}\eta_{\beta\epsilon} + g_{\beta\epsilon}\eta_{\alpha\delta} - g_{\beta\delta}\eta_{\alpha\epsilon} + x_\delta\partial_\epsilon g_{\alpha\beta} - x_\epsilon\partial_\delta g_{\alpha\beta} = 0. \quad (2)$$

The metric components $g_{\alpha\beta}$ are components of a covariant second rank tensor under the Lorentz transformations. They must be constructed from the Lorentz metric $\eta_{\alpha\beta}$ and point coordinates $x = \{x^\alpha\}$. The only possibility is the metric of the form

$$g_{\alpha\beta} = A\eta_{\alpha\beta} + Bx_\alpha x_\beta,$$

where A and B are some functions on $\mathbb{R}^{1,n-1}$. The substitution of this metric into the Killing equations (2) restricts the form of the functions A and B . One may prove that they can be arbitrary functions of only one variable

$$s := x^\alpha x^\beta \eta_{\alpha\beta},$$

which is invariant under the Lorentz transformations. Thus, the metric invariant with respect to the Lorentz transformations is parameterized by two arbitrary functions $A(s)$ and $B(s)$. It is useful to rewrite it in another form

$$g_{\alpha\beta} = f(s)\Pi_{\alpha\beta}^T + g(s)\Pi_{\alpha\beta}^L = f\eta_{\alpha\beta} + (g - f)\frac{x_\alpha x_\beta}{s}, \quad (3)$$

where Π^T and Π^L are projection operators:

$$\Pi_{\alpha\beta}^T := \eta_{\alpha\beta} - \frac{x_\alpha x_\beta}{s}, \quad \Pi_{\alpha\beta}^L := \frac{x_\alpha x_\beta}{s}.$$

The determinant of the metric (3) is easily computed:

$$\det g_{\alpha\beta} = (-f)^{n-1}g. \quad (4)$$

Therefore, the Lorentz invariant metric is degenerate if and only if $fg = 0$. We suppose that functions f and g are sufficiently smooth, and $f > 0$ and $g \neq 0$. Moreover we suppose that there exists a limit

$$\lim_{s \rightarrow 0} \frac{f(s) - g(s)}{s},$$

which is necessary for the metric to be defined for $s = 0$.

We use the Lorentz metric $\eta_{\alpha\beta}$ to raise and lower indices unless otherwise stated.

The Lorentz invariant metric (3) for $f = g$ was considered by V. A. Fock [?].

The invariant interval

$$ds^2 = f dx_\alpha dx^\alpha + (g - f) \frac{(x_\alpha dx^\alpha)^2}{s}$$

corresponds to the metric (3).

The metric tensor (3) has the same form in all coordinate systems related by Lorentz transformations. However, it changes under translations $x^\alpha \mapsto x^\alpha + a^\alpha$ because the metric depends explicitly on coordinates and the origin of the coordinate system is distinguished.

The expression for the metric (3) in terms of projection operators is useful because the inverse metric has a simple form

$$g^{\alpha\beta} = \frac{1}{f}\Pi^{T\alpha\beta} + \frac{1}{g}\Pi^{L\alpha\beta}. \quad (5)$$

For $g > 0$, the vierbein

$$e_\alpha^a = \sqrt{f}\delta_\alpha^a + (\sqrt{g} - \sqrt{f})\frac{x_\alpha x^a}{s}. \quad (6)$$

can be attributed to the metric (3).

One can easily check the following properties of the projection operators:

$$\begin{aligned} \Pi^{\text{T}\alpha\beta}x_\beta &= 0, & \Pi_\alpha^{\text{T}\alpha} &= n-1, & \partial_\alpha \Pi_{\beta\gamma}^{\text{T}} &= -\frac{\Pi_{\alpha\beta}^{\text{T}}x_\gamma + \Pi_{\alpha\gamma}^{\text{T}}x_\beta}{s}, \\ \Pi^{\text{L}\alpha\beta}x_\beta &= x^\alpha, & \Pi_\alpha^{\text{L}\alpha} &= 1, & \partial_\alpha \Pi_{\beta\gamma}^{\text{L}} &= \frac{\Pi_{\alpha\beta}^{\text{T}}x_\gamma + \Pi_{\alpha\gamma}^{\text{T}}x_\beta}{s}, \end{aligned}$$

which will be used in calculations below.

Simple calculations yield Christoffel's symbols for the metric (3)

$$\Gamma_{\alpha\beta}^\gamma = \frac{f'}{f}(x_\alpha \Pi_\beta^{\text{T}\gamma} + x_\beta \Pi_\alpha^{\text{T}\gamma}) + \frac{g'}{g}(x_\alpha \Pi_\beta^{\text{L}\gamma} + x_\beta \Pi_\alpha^{\text{L}\gamma} - x^\gamma \Pi_{\alpha\beta}^{\text{L}}) + \frac{g-f-f's}{sg}x^\gamma \Pi_{\alpha\beta}^{\text{T}}, \quad (7)$$

where the prime denotes differentiation with respect to s . The curvature tensor for the metric (3) is

$$\begin{aligned} R_{\alpha\beta\gamma}^\delta &= \Pi_{\alpha\gamma}^{\text{T}} \Pi_\beta^{\text{T}\delta} \left[\frac{(f+f's)^2}{sfg} - \frac{1}{s} \right] + \\ &+ \Pi_{\alpha\gamma}^{\text{L}} \Pi_\beta^{\text{T}\delta} \left[2 \left(\frac{f+f's}{f} \right)' + \left(\frac{f'}{f} - \frac{g'}{g} \right) \frac{f+f's}{f} \right] + \\ &+ \Pi_\alpha^{\text{L}\delta} \Pi_{\beta\gamma}^{\text{T}} \left[-2 \left(\frac{f+f's}{g} \right)' + \left(\frac{f'}{f} - \frac{g'}{g} \right) \frac{f+f's}{g} \right] - (\alpha \leftrightarrow \beta), \end{aligned} \quad (8)$$

where $(\alpha \leftrightarrow \beta)$ stands for the preceding terms with the exchanged indices. Contracting this expression in the indices β and δ yields the Ricci tensor

$$\begin{aligned} R_{\alpha\beta} &= \Pi_{\alpha\beta}^{\text{T}} \left[\frac{n-2}{s} \left(\frac{(f+f's)^2}{fg} - 1 \right) + 2 \frac{(f+f's)'}{g} - \left(\frac{f'}{f} + \frac{g'}{g} \right) \frac{f+f's}{g} \right] + \\ &+ \Pi_{\alpha\beta}^{\text{L}}(n-1) \left[2 \frac{(f+f's)'}{f} - \left(\frac{f'}{f} + \frac{g'}{g} \right) \frac{f+f's}{f} \right]. \end{aligned} \quad (9)$$

The further contraction with the inverse metric (5) gives the scalar curvature

$$R = (n-1) \left[\frac{n-2}{fs} \left(\frac{(f+f's)^2}{fg} - 1 \right) + 4 \frac{(f+f's)'}{fg} - 2 \left(\frac{f'}{f} + \frac{g'}{g} \right) \frac{f+f's}{fg} \right]. \quad (10)$$

Constant curvature spaces defined by the equation

$$R_{\alpha\beta\gamma\delta} = -\frac{2K}{n(n-1)}(g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \quad (11)$$

with some constant K automatically satisfy the vacuum Einstein's equations with cosmological constant. Let us solve equation (11) for the Lorentz invariant metric (3). To this end, we lower the last index of the curvature tensor (8) using the metric (3)

$$\begin{aligned} R_{\alpha\beta\gamma\delta} &= \Pi_{\alpha\gamma}^{\text{T}} \Pi_{\beta\delta}^{\text{T}} \frac{1}{s} \left[\frac{(f+f's)^2}{g} - f \right] + \\ &+ (\Pi_{\alpha\gamma}^{\text{L}} \Pi_{\beta\delta}^{\text{T}} - \Pi_{\alpha\delta}^{\text{L}} \Pi_{\beta\gamma}^{\text{T}}) \left[2(f+f's)' - \left(\frac{f'}{f} + \frac{g'}{g} \right) (f+f's) \right] - (\alpha \leftrightarrow \beta) \end{aligned} \quad (12)$$

and substitute it into equation (11). As a result, we get the system of differential equations for the functions f and g :

$$\frac{(f + f's)^2}{sg} - \frac{f}{s} = -\frac{2K}{n(n-1)}f^2, \quad (13)$$

$$2(f + f's)' - \left(\frac{f'}{f} + \frac{g'}{g}\right)(f + f's) = -\frac{2K}{n(n-1)}fg. \quad (14)$$

The first equation yields a solution for the function g :

$$g = \frac{(f + f's)^2}{f \left(1 - \frac{2K}{n(n-1)}fs\right)}. \quad (15)$$

Since the inequality $g \neq 0$ must be fulfilled for the metric to be nondegenerate, the function f should satisfy the inequality

$$f \neq \frac{n(n-1)}{2Ks}, \quad s \neq 0. \quad (16)$$

The substitution of expression (15) into the second equation (14) yields the identity. Thus, we have proved the first part of the following statement:

Theorem 0.1. *The Lorentz invariant metric*

$$g_{\alpha\beta} = f\Pi_{\alpha\beta}^T + \frac{(f + f's)^2}{f \left(1 - \frac{2K}{n(n-1)}fs\right)}\Pi_{\alpha\beta}^L, \quad (17)$$

where $f(s)$ is an arbitrary positive function satisfying equation (16), is the metric of a constant curvature space. Conversely, the metric of a constant curvature space can be written in the Lorentz invariant form (17) for some function $f(s)$.

Proof. It remains to prove that any metric of a constant curvature space can be transformed into the Lorentz invariant form (3). To show this, we write the metric (17) in a more familiar form. To this end we fix the function f by putting $f = g$ in the initial representation (3). Then equation (15) yields the differential equation

$$f'^2s + 2f'f + \frac{2K}{n(n-1)}f^3 = 0,$$

which has a general solution

$$f = \frac{C}{\left(C + \frac{K}{2n(n-1)}s\right)^2}, \quad C = \text{const.}$$

The integration constant can be removed by rescaling coordinates. Therefore, we put $C = 1$ without loss of generality. As a result we obtain the metric of constant curvature

$$g_{\alpha\beta} = \frac{\eta_{\alpha\beta}}{\left(1 + \frac{K}{2n(n-1)}s\right)^2}. \quad (18)$$

The fact that any constant curvature metric can be reduced to this form is well known. The proof of this fact is nontrivial (see, e.g. [5]). \square

The performed calculations can be easily transferred to the Euclidean space metric which is invariant with respect to $\mathbb{SO}(n)$ rotations. In order to do this, one should replace the Lorentzian metric $\eta_{\alpha\beta}$ by the Euclidean metric $\delta_{\alpha\beta}$ in all formulas.

Since

$$\lim_{s \rightarrow 0} \frac{g - f}{s} = 2f' + \frac{2K}{n(n-1)} f^2,$$

the expression for the metric (17) is defined for $s = 0$ as well.

Now we solve the vacuum Einstein equations with the cosmological constant

$$R_{\alpha\beta} = \Lambda g_{\alpha\beta}$$

for a Lorentz invariant metric. One could expect that these equations have solutions not only of constant curvature because their number is less then the number of equations in the constant curvature condition (11). However, for Lorentz invariant metrics the classes of solutions coincide. Indeed, the substitution of the Ricci tensor (9) into Einstein's equations yields the following system of equations:

$$\frac{n-2}{s} \left[\frac{(f + f's)^2}{fg} - 1 \right] + 2 \frac{(f + f's)'}{g} - \left(\frac{f'}{f} + \frac{g'}{g} \right) \frac{f + f's}{g} = \Lambda f, \quad (19)$$

$$(n-1) \left[2 \frac{(f + f's)'}{f} - \left(\frac{f'}{f} + \frac{g'}{g} \right) \frac{f + f's}{f} \right] = \Lambda g. \quad (20)$$

The second equation coincides with equation (14) for

$$\Lambda = -\frac{2K}{n}.$$

The linear combination of equations (19) and (20) with the coefficients $1/f$ and $-1/g$, respectively, is equivalent to equation (13).

Thus, we proved that all Lorentz invariant solutions of the vacuum Einstein equations with cosmological constant are exhausted by constant curvature spaces.

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